Sharp Thresholds for Overlap Gap Property: Ising p-Spin Glass and Random k-SAT

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Ising p-Spin Glass

For $p, N \in \mathbb{N}$ and **disorder** $\boldsymbol{J} = (J_{i_1,\dots,i_p} : 1 \le i_1, \dots, i_p \le N)$ with iid $\mathcal{N}(0,1)$ entries, consider Hamiltonian

$$H_{N,p}(\boldsymbol{\sigma}) := N^{-\frac{p-1}{2}} \sum_{1 \le i_1,\dots,i_p \le N} J_{i_1,\dots,i_p} \boldsymbol{\sigma}_{i_1} \cdots \boldsymbol{\sigma}_{i_p},$$

where $\boldsymbol{\sigma} \in \{-1, 1\}^N$ (Ising spins).

- $p = 2$ by Sherrington-Kirkpatrick [75]. Large p by Derrida [80]
- MLE in tensor PCA, limits of MaxCUT/MaxSAT
  Ben Arous-Mei-Montanari-Nica [19], Dembo-Montanari-Sen [18], Panchenko [18]

Optimizing Ising p-Spin Glass

Scaling $N^{-\frac{p-1}{2}}$ ensures non-trivial limit

$$\mathrm{OPT} := \lim_{N\to\infty} \max_{\boldsymbol{\sigma}\in\{-1,1\}^N} H_{N,p}(\boldsymbol{\sigma})/N = \Theta(1)$$

Parisi [79], Guerra-Toninelli [02], Talagrand [06], Panchenko [14]

Algorithmic Problem

Given $\boldsymbol{J}$, find in **poly-time** $\boldsymbol{\sigma}_{\mathrm{ALG}}$ st $H_{N,p}(\boldsymbol{\sigma}_{\mathrm{ALG}})/N \geq (1-\epsilon)\mathrm{OPT}$.

- H_N **non-convex** and $e^{\Theta(N)}$ local minima/saddle below OPT
  Auffinger-Ben Arous-Cerny [13], Subag [17], Subag-Zeitouni [21]
- **Worst-Case**: Reaching $H(\boldsymbol{\sigma})/N \geq \mathrm{OPT} \times \log^{-c} N$ is **NP-hard**
  Arora-Berger-Hazan-Kindler-Safra [05]

Statistical-Computational Gap in Spin Glasses

- For SK model ($p = 2$), AMP succeeds under **no overlap-gap** assumption (Montanari [19])

- For $p \geq 4$, model exhibits **overlap-gap** (Chen-Gamarnik-Panchenko-Rahman [19]). AMP **fails**:

Theorem (Gamarnik-Jagannath, 2021)

$\forall p \geq 4$ *even,* $\exists \mu_p > 0$ *st for* **AMP**, $H_{N,p}(\boldsymbol{\sigma}_{\mathrm{ALG}})/N \leq \mathrm{OPT} - \mu_p$.

Gap: $\mathrm{ALG} < \mathrm{OPT}$

No Overlap-Gap Assumption

No Overlap Gap Assumption

μ_β : distribution of $n^{-1}|\langle \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \rangle|$. For $\beta > \beta_0$, $\exists q^*$ st $t \mapsto \mu_\beta([0,t])$ is strictly increasing on $[0, q^*]$ with $\mu_\beta([0, q^*]) = 1$

- Set of achievable overlaps for typical low-temperature states is an **interval**.
- **Open**, though **widely believed** for the **SK model**.

Random k-SAT

- Boolean variables $x_1, \ldots, x_N$. k-clause $\mathcal{C} = y_1 \vee \cdots \vee y_k$, where $y_1, \ldots, y_k$ chosen from $\{x_1, x'_1, \ldots, x_N, x'_N\}$.
- **Formula:** $\Phi = \mathcal{C}_1 \wedge \cdots \wedge \mathcal{C}_M$, for iid $\mathcal{C}_i$
- **Regime:** $M = \Theta(N)$, constraint density $\alpha := M/N$.

When do satisfying $\boldsymbol{\sigma}$ exist? Can we find them efficiently?

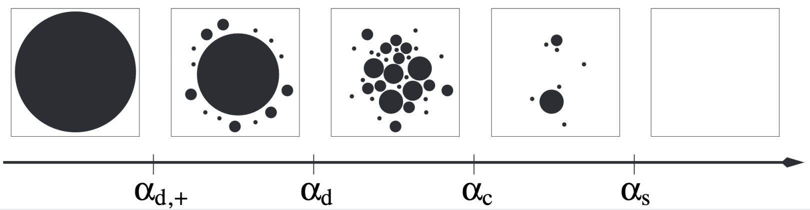
Random k-SAT: $\alpha_{\mathrm{SAT}} \sim 2^k \ln 2$

- Φ is whp unsatisfiable if $\alpha \geq 2^k \ln 2 - \frac{1}{2}(\ln 2 + 1) + o_k(1)$
  Franco-Paull [83], Kirousis-Kranakis-Krizanc-Stamatiou [98]
- Φ is whp satisfiable if $\alpha \leq 2^k \ln 2 - \frac{1}{2}(\ln 2 + 1) - o_k(1)$
  Achlioptas-Moore [02], Achlioptas-Peres [03], Coja Oghlan-Panagiotou [16]
- For $k \geq k_0$, $\exists \alpha_{\mathrm{SAT}}(k)$ sth Φ is satisfiable if $\alpha < \alpha_{\mathrm{SAT}}(k)$ and unsatisfiable if $\alpha > \alpha_{\mathrm{SAT}}(k)$, both whp.
  Ding-Sly-Sun [15]

Statistical-Computational Gap in Random k-SAT

Algorithms work below $\alpha \leq \alpha_{\mathrm{ALG}} \sim 2^k \ln k/k$ (Coja-Oghlan [10]).

Shattering at $\alpha_d \sim \alpha_{\mathrm{ALG}}$



Krzakala-Montanari-Ricci Tersenghi-Zdeborova [07], Achlioptas-Coja-Oghlan [08]

- Low-degree poly fail above $\alpha \geq 4.91\alpha_{\mathrm{ALG}}$ (Bresler-Huang [21])
- Prior hardness results for $\alpha = \Omega(2^k \ln^2 k/k) = \Omega(\alpha_{\mathrm{ALG}} \ln k)$
  Gamarnik-Sudan [17], Coja Oglan-Haqshenas-Hetterich [17]

Overlap Gap Property

Both models exhibit **Overlap Gap Property** (OGP), intricate geometric feature of landscape. Implies hardness of stable algs

Gamarnik-Jagannath [19], Gamarnik-Jagannath-Wein [20], Huang-Sellke [22,23], Gamarnik-Jagannath-**K.** [23], Gamarnik-Sudan [17], Bresler-Huang [21]

Stable Algorithms

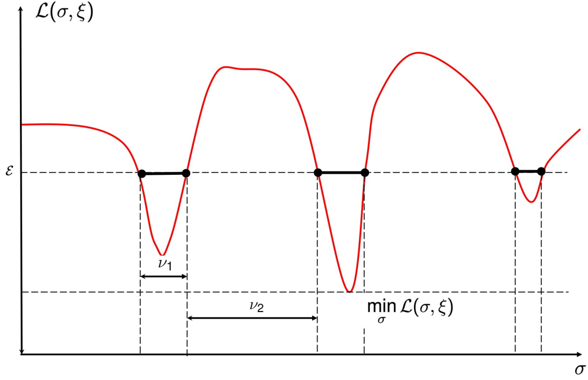
AMP/BP, low-degree poly, spectral algs, GD/Langevin dynamics

OGP: No tuples of near-optimal solutions at intermediate overlap.

OGP $\Rightarrow$ Algorithmic Hardness

OGP - A Cartoon Illustration

Loss $\mathcal{L}$. No $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ with $\mathcal{L}(\boldsymbol{\sigma}_i) \le \mathcal{E}, d(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) \in (\nu_1, \nu_2)$.



$\nu_1 < \nu_2$

Finer Link between OGP and Hardness?

- Algs succeed for **SK model** under no **overlap gap** assumption (Montanari [19])
- For binary perceptron, OGP guarantee (Gamarnik-**K.**-Perkins-Xu [22]) matches usual average-case hardness (Vafa-Vaikuntanathan [25])
- Certain optimization problems exhibiting OGP are amenable to linear programming (Li-Schramm [24])

$$\textbf{OGP} \overset{?}{\Longleftrightarrow} \textbf{Hardness}$$

OGP to Algorithmic Hardness

Rule-of-Thumb

Multi OGP (m-OGP): larger m gives better thresholds (first moment)

How does the power of OGP scale as m increases?

Our Results

- Sharp phase transition for **multi OGP** for large p and k
- Qualitative insights into power of OGP: power indeed amplifies with growing m

OGP in Spin Glasses

- p-spin model exhibits **branching OGP**, Lipschitz algs fail at $(1+\epsilon)\mathrm{ALG}$. Sharpest bound for fixed p Huang-Sellke [22,23]
- **Lipschitz Alg:** $O(1)$ iterations of AMP, Gradient Descent, Langevin Dynamics (on $e^{\beta H}$). Strict subclass of stable alg
- Valid only for **even** p, sophisticated pf (Parisi formula)

Ground-State Asymptotics

As $p \to \infty$, p-spin glass converge to REM where $\mathrm{OPT} = \sqrt{2 \ln 2}$

Symmetric m-OGP

For $m \in \mathbb{N}$, $0 < \gamma < 1$ and $0 < \eta \ll \xi < 1$, let $\mathcal{S}_{\mathrm{p-spin}}$ be the set of all m-tuples $\boldsymbol{\sigma}^{(1)}, \dots, \boldsymbol{\sigma}^{(m)} \in \{-1, 1\}^N$ such that:

- $\min_{1 \le t \le m} H_{N,p}(\boldsymbol{\sigma}^{(t)}) \ge \gamma\sqrt{2 \ln 2}$
- For any $t < \ell$, $n^{-1}\langle \boldsymbol{\sigma}^{(t)}, \boldsymbol{\sigma}^{(\ell)} \rangle \in [\xi - \eta, \xi]$.

Symmetric m-OGP

$\mathcal{S}_{\mathrm{p-spin}} = \varnothing$ (whp) for suitable parameters.

No nearly equidistant, γ-optimal m-tuples

Symmetric m-OGP

Theorem (Gamarnik-Jagannath-K., 2023)

$\forall m \in \mathbb{N}, \gamma > 1/\sqrt{m}$ and $p = \Omega(1)$, the model exhibits symmetric m-OGP and stable algorithms fail.

Asymptotically Sharp (Addario Berry-Maillard [20])

Finding $H_N(\boldsymbol{\sigma}_{\mathrm{ALG}})/N \ge \epsilon$ in REM requires $e^{\Theta(N)}$ queries

Theorem (K., 2025)

Symmetric m-OGP is absent below $1/\sqrt{m}$: $\forall m \in \mathbb{N}, \gamma < 1/\sqrt{m}$, $0 < \eta < \xi < 1$ and p large, $\mathcal{S}_{\mathrm{p-spin}} \ne \varnothing$ (whp)

Sharp Phase Transition for m-OGP

Corollary

$\forall m \in \mathbb{N}$, symmetric m-OGP exhibits sharp PT at $1/\sqrt{m}$

Power of symmetric m-OGP in proving hardness amplifies indefinitely

Shattering

- Kirkpatrick-Thirumalai [87]: Ising p-spin glass exhibits shattering phase. Verified for $\sqrt{\ln 2} < \beta < \sqrt{2 \ln 2}$ Gamarnik-Jagannath-**K.** [23]
- Crucially based on 2-OGP, absent for $\gamma < 1/\sqrt{2}$. $\beta < \sqrt{\log 2}$?
- El Alaoui [24]: Soft OGP, optimal shattering for $\beta > \sqrt{2 \log p / p}$

OGP in Random k-SAT

For $k, m \in \mathbb{N}$, $0 < \gamma < 1$ and $0 < \eta \ll \xi < 1$, let $\mathcal{S}_{\mathrm{k-SAT}}$ be the set of all m-tuples $\boldsymbol{\sigma}^{(1)}, \dots, \boldsymbol{\sigma}^{(m)} \in \{0,1\}^N$ such that:

- $\Phi(\boldsymbol{\sigma}^{(t)}) = 1, \forall t$, where **density** is $\gamma 2^k \ln 2$
- For any $t < \ell$, $N^{-1} d_H(\boldsymbol{\sigma}^{(t)}, \boldsymbol{\sigma}^{(\ell)}) \in [\xi - \eta, \xi]$.

Symmetric m-OGP

$\mathcal{S}_{\mathrm{k-SAT}} = \varnothing$ (whp) for suitable parameters.

No nearly equidistant satisfying m-tuples at density $\sim \gamma \alpha_{\mathrm{SAT}}$

OGP in Random k-SAT

Theorem (K., 2025)

- $\forall m, \gamma > 1/m$, and $k = \Omega(1)$, *model exhibits symmetric m-OGP*
- $\forall m, \gamma < 1/m$, and $k = \Omega(\ln n)$, *symmetric m-OGP is absent*

Corollary

$\forall m$, *symmetric m-OGP exhibits sharp PT at* $1/m$.

$k = \Omega(\ln n)$: Mirrors earlier results from the random k-SAT literature
Frieze-Wormald [05], Coja Oghlan-Frieze [80]

How about $k = O(1)$?

Constant k

Consider m-tuples satisfying fraction of clauses (MaxSAT)

Theorem (K., 2025)

$\forall m, \gamma < 1/m$ and $k = \Omega(1)$, symmetric m-OGP is absent from the set of assignments satisfying $1 - \exp\big(-\Theta_k(k)\big)$ fraction of $\mathcal{C}$

Proof Idea: Second Moment Method

- Using **probabilistic method**, $\exists \boldsymbol{\sigma}^{(1)}, \dots, \boldsymbol{\sigma}^{(m)}$ st for all $i < j$, $N^{-1}\langle \boldsymbol{\sigma}^{(i)}, \boldsymbol{\sigma}^{(j)} \rangle \in [\xi - \eta, \xi]$ (otherwise $\mathcal{S}_{\mathrm{p-spin}} = \varnothing$ trivially).

- **Second moment method:** For any $\mathbb{Z}$-valued $M \geq 0$,

$$\mathbb{P}[M \geq 1] \geq \frac{\mathbb{E}[M]^2}{\mathbb{E}[M^2]} \quad \text{(Paley-Zygmund Inequality)}$$

  Applying it to $M := |\mathcal{S}_{p-spin}|$ we obtain

$$\mathbb{P}\left[\mathcal{S}_{p-spin} \neq \varnothing\right] \geq \exp\big(-N o_p(1)\big)$$

  where $o_p(1) \to 0$ as $p \to \infty$. **2nd Moment Method fails!**

Proof Idea: Concentration + Repairing 2nd Mom

Proxy random variable

$$T_{m,\xi,\eta} := \max_{\substack{\boldsymbol{\sigma}^{(1)},\ldots,\boldsymbol{\sigma}^{(m)} \in \{-1,1\}^N \\ N^{-1}\langle \boldsymbol{\sigma}^{(i)},\boldsymbol{\sigma}^{(j)}\rangle \in [\xi-\eta,\xi]}} \min_{1\le j\le m} H_{N,p}(\boldsymbol{\sigma}^{(i)})$$

Lipschitz wrt **disorder**. As $\boldsymbol{J} \in (\mathbb{R}^N)^{\otimes p}$ has iid $\mathcal{N}(0,1)$ entries,

$$\mathbb{P}\big[\big|T_{m,\xi,\eta} - \mathbb{E}[T_{m,\xi,\eta}]\big| \ge \epsilon\big] \le 2\exp\big(-n\epsilon^2\big)$$

Observe that:

$$\mathcal{S}_{\mathrm{p-spin}} \ne \varnothing \iff T_{m,\xi,\eta} \ge \gamma\sqrt{2\ln 2}$$

Proof Idea: An Argument of Frieze

Fix $\gamma' \in (\gamma, 1/\sqrt{m})$. For all p, N large,

$$\mathbb{P}[T_{m,\xi,\eta} \geq \gamma' \sqrt{2 \ln 2}] \geq \exp\big(-N o_p(1)\big)$$
$$\geq 2 \exp\big(-N\epsilon^2\big) \geq \mathbb{P}[T_{m,\xi,\eta} \geq \mathbb{E}[T_{m,\xi,\eta}] + \epsilon].$$

Consequently, $\mathbb{E}[T_{m,\xi,\eta}] \geq \gamma' \sqrt{2 \ln 2} - \epsilon$. Thus for $\epsilon > 0$ small

$$T_{m,\xi,\eta} \geq \gamma' \sqrt{2 \ln 2} - 2\epsilon \geq \gamma \sqrt{2 \ln 2} \quad \text{whp.}$$

- Argument of Frieze [90], originally for indep sets in $\mathbb{G}(n, \frac{d}{n})$
- k-SAT: Auxil rv has bdd differences, **McDiarmid's ineq**

Thank you!